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ABSTRACT

In a study of mathematics learning, taking a rule-oriented approach, students were taught to trade objects of type A for objects of types B and C. Children ranging from ages 7 to 9 were given rules for converting A to B and B to C and were then presented with the task of converting A to C. Of the 30, 6 succeeded. Of the 24 who failed, half were taught via a higher order rule to link arbitrary pairs of compatible rules. The other 12 pupils were not given this training. Then, all of the students were presented with a new pair of compatible rules for converting A to B and B to C. Finally, they were presented with a corresponding A to C task. The results showed that all of the students who received training succeeded whereas not one of the other students did so. The results, it is stated, indicated that knowing the components of a solution (rule) for a task is not sufficient. Integrating components is accomplished by applying higher order rules to lower order ones (e.g., component rules). Component rules and higher order rules, although behaviorally sufficient for problem solving, are not logically sufficient. Presumably, it is asserted, the learner must have some innate capability which tells him when and how the various rules are to be used in attacking a problem. With regard to the last point, an innate mechanism which is basic to a structural learning theory is then discussed. (Author/JM)

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HOW DOES MATHEMATICS LEARNING TAKE PLACE? *

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How does learning take place? This is a question that has puzzled generations in mathematics education and goes back to antiquity. Indeed, the question persists today. At a recent structural learning conference sponsored by the MERGE Research Institute in Philadelphia, a number of world education leaders discussed this very issue. There were on the panel leading representatives from artificial intelligence, computer science, experimental and mathematical psychology, developmental psychology, mathematics education, educational technology, educational psychology, mathematical logic and philosophy. I am sure that most readers will be happy to know that not one person defended the old behaviorist point of view -- no one seriously proposed that people have bodies for responding but no mind to tell the body what to do.

Beyond that point of agreement, however, there were major differences. Those with a computer orientation tended to view knowledge as consisting of programs, or equivalently, as rules or procedures. In this view, learning consists of some form of modifying or adding new procedures. The developmental psychologists, given their concern with changes in behavior over long periods of time, tended to question the very notion of learning in the usual short-term sense. The mathematics educators, in part, tended toward general descriptions of the stages children seem to go through while learning, but said little about the more basic question of why and how this takes place.

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Those who have been most closely associated with the laboratory study of learning, the experimental psychologists, oddly enough, had the least to say.

Among those with some conviction on the subject, the general consensus seemed to lean toward the rule-oriented view of knowledge, with cautions to the effect that one must insure that long term effects are included in the picture.

Having been one of the more optimistic members of the panel, let me tell you a little about my views. I should like to begin, however, with two caveats: (1) What I shall have to say is just one small part of what by now is a rather comprehensive theory of complex human functioning. This theory takes into account not only the learner but the teacher as well. (Those of you who may be interested in learning more about the theory can write me at the University of Pennsylvania.) (2) Although I have very definite ideas as to how learning takes place, and although I have stated these views in my book (Scandura, 1973) with considerable precision, answering the question of how learning takes place in full generality is very difficult. This question is at once the most important and most difficult question to answer about human behavior. Furthermore, the more precisely one attempts to answer the question, the more difficult it becomes.

Let me begin with a very simple problem, one of the simplest that we as mathematics educators might be concerned with. We present the child with the task of converting 5 yards into inches. If we do this, some children will succeed, but others will fail. Why? Do those who succeed do so because they know something that the other children don't? Or, do they succeed because they are inherently superior individuals? Or, perhaps both factors enter to some extent. But, to what extent, and how?

Let us consider how a subject might succeed on this simple task. One possibility would be to simply know a rule for converting yards into inches. -- The rule "multiply by 36" would do. But, is knowing this rule sufficient? Even the slowest

child has during his lifetime learned a wide variety of rules. On what basis does he decide to use this one?

Let me outline a second possibility -- that the subject does not know explicitly a rule for converting between yards and inches. Yet, even in this type of situation, many children will succeed. Suppose, for example, that they have already learned, or that we teach them, a rule for converting between yards and feet and another rule, between feet and inches. And, then, we present the task of converting 5 yards into inches. Will the children succeed? How many will succeed?

With these questions as background, let me describe an experiment that we have conducted that poses a problem very much like that in converting between yards and inches. The students were taught how to trade objects of one kind for objects of another kind.

Insert Figure 1

The card at the top of Figure 1, for example, denotes a rule for exchanging paper clips for blue chips. In particular, if 2 paper clips are presented to the pupil, he is to trade for 3 blue chips; similarly, if presented with 3 paper clips, he is to trade for 4 blue chips. The composite card at the bottom denotes a composite rule in which trades are made in two steps. In the particular example shown, the subject first trades paper clips for pencils by adding 2 extra paper clips and then white chips for the paper clips by adding 1 extra white chip. For discussion purposes, compatible simple rules are denoted A→B and B→C whereas corresponding composite rules are denoted A→B→C.

Given tasks of this type, suppose we teach a learner an A→B and a B→C rule, and then present him with a certain number of A objects, and ask him to trade for the appropriate number of C objects. Under these conditions: (1) How many children will succeed on the A→?C task? If some of the children fail, what else do they need to know in order to succeed? (3) Is there anything we can take for granted about the capabilities of all children -- something which is basic, which is innate to all children, say, of the ages of 7 and above?

In an experiment conducted with children ranging from ages 7-9, we found that only 6 of 30 children were able to solve a new A→?C task of this type without explicit training on an underlying A→C rule. Of the 24 who failed, half were taught a higher order rule by which arbitrary pairs of compatible rules, in which the output of one may serve as the input of the other, could be combined to form composite rules. The other 12 pupils were not given this training. Then, all of the students were presented with a new pair of compatible rules, A'→B' and B'→C', rules none of the children had seen before. Finally, they were presented with the corresponding A'→?C' task.

The results showed that all of the subjects who received training succeeded on the A'→?C task whereas not one of the other students did so.

Several things whould be observed about this study. (1) Knowing the components of a solution (rule) for a task is not sufficient. The subject must know how to put the components together appropriately. (2) Integrating components is accomplished by applying higher order rules to lower order ones (e.g., component rules). Indeed, higher order rules seem to be both a necessary and a sufficient condition for solving problems where needed components are available. (3) Component rules and higher order rules, although behaviorally sufficient for problem solving, are not logically sufficient. Presumably, the learner must have some innate capability which tells him how and when the various rules are to be used in attacking a problem.

With regard to the third point, let us consider an innate mechanism which is basic to my structural learning theory (Scandura, 1973). This mechanism tells how known rules interact in learning and performance and may be expressed in terms of three simple hypotheses. Hyp. 1 (simple performance hypothesis): Given a goal and the availability of one or more rules, each of which generates the desired response, then the subject will use one of them. Hyp. 2 (control shift hypothesis): If the subject does not have a rule immediately available for achieving his goal, control automatically shifts to the higher level goal of deriving such a rule. Hyp. 3 (learning and reversion hypothesis): Once a higher level goal is satisfied, the newly derived rule is added to available knowledge (i.e., is learned) and control reverts back to the original goal.

To see how this mechanism operates, consider the task of converting A objects into C objects, and two students, one who enters a task knowing a rule for converting A objects into C objects and another who only knows rules for converting A objects into B objects, B objects into C objects, and the higher order composition rule which operates on pairs of compatible rules and forms composite ones.

Is our postulated control mechanism sufficient for predicting the performance of these subjects? The success of the first student (on A→?C) follows directly from the simple performance hypothesis. An available rule applies in the task situation, so he, therefore, uses it. The situation with the second student is only slightly more involved. The subject first checks his available rules, and since none applies, control shifts to the higher level goal of generating one. In this higher level goal situation, the higher order composition rule applies, so it is used (i.e., applied to the A→B and B→C rules) to generate the composite rule A→B→C. This composite rule satisfies the higher level goal; therefore, the A→B→C rule is added to the available knowledge base and control reverts to the original goal. At this point, the newly derived rule applies, so according to the simple performance hypothesis, it is used and the problem is solved.

Could the answer to how learning takes place be all that simple? Not quite. There is more to be said about such things as processing time, how much information a person can keep in mind at the same time, and so on. But, for present purposes we can ignore these factors. This simple mechanism does account for a "whole lot." Consider, for example, discovery learning. How do children do it? Surely, that is too complex to yield to such a simple idea! Or, is it?

Suppose we present a child with the number 1 and ask him to guess what number goes with it. Presumably, he guesses wildly, and we tell him that the correct response is 3. Next, we present 2 and he again guesses incorrectly. We tell him that the correct number is 6. We do the same with the number 3, to which the correct response is 9. At this point the child says, "I've got it." To check him, we present the number 8 and he gives the correct response, 24. How does he manage to do this?

According to our previous analysis, it must be that the learner has derived a solution rule by use of some higher order rule. Indeed, this is precisely what I believe has happened. The rule shown in Figure 2 operates on three numbered pairs

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Insert Figure 2

(which may be thought of as three degenerate rules) and generates the rule "multiply by 3." Not only is this higher order generalization rule sufficient for generating "discoveries" (i.e., solution rules), but it seems to be highly consistent with human performance. In a study similar to the one I have just described, we tested children in a remedial junior high mathematics program to see which kinds of discoveries they were able to make without training. Then, we presented them with higher order rules which could be used to make discoveries in

predetermined task situations and tested them on new tasks. In every case, the students succeeded if and only if, either they already knew an appropriate higher order rule, which applied in the given situation, or they had been explicitly taught one. (For more details, see Scandura, 1973a.)

From this study, we can conclude two things: (1) Higher order rules for making discoveries do seem to exist. (2) The presumed control mechanism also seems adequate in explaining discovery learning.

As simple as it is, we have hardly begun to tap the potential of this simple learning mechanism. However, space will not allow us more than a brief overview of a few additional areas of application. First, let's consider the question of the cumulative effects of learning. Why does learning sometimes progress more efficiently when information and tasks are presented in one order rather than in another?

Suppose, for example, that two children enter a learning situation with the following available knowledge: A→B, B→C, C→D, and $\Rightarrow$ (a higher order composition rule). We then present the tasks of trading A objects for C objects (A→ ?C) and A objects for D objects (A→ ?D), in both possible orders.

Task Sequence A	Task Sequence B
(1) A → (?D)	(1) A →(?C)
Failure	Success
	Rule A→ B→C Learned*
(2) A → (?C)	(2) A→ (?D)
Success	Success
Rule A→ B→C Learned*	Rule A→B→C→D Learned*

* Added to available rule set.

When the A→?D task is presented first, the subject fails. The higher order rule applies only to pairs of rules; to succeed in this case, it would have to apply to triples. But, when task A→?D is presented second, as in task sequence B, the subject succeeds. The reason for success, in this case, is that the subject has

first learned rule A→B→C in the process of solving task A→?C.

Let me now remark just briefly on one further matter that may be causing the reader some concern. Do I really think that people only learn large sets of discrete rules?

To the contrary, the rules which bring about or enter into learning may be rather comprehensive in scope -- and as a result of learning, they may grow to become even more so. Indeed, even our simple trading game analysis shows how available information may be combined to form larger units.

On the other hand, there is nothing in the model which requires that learning be comprehensive in all cases. The theory is neutral in this regard as are all theories in natural science. Science does not make moral judgements; only men do so.

As an example, let us consider the learning of traditional trigonometric

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Insert Figure 3

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identities. Figure 3 represents the knowledge that might be available to two different students, A and B. You will recognize each of them immediately. Student A has memorized the three identities and perhaps has learned to use them in solving routine problems. Student B, on the other hand, may not only have learned the identities but how they relate to a variety of other bits of information he or she may have learned about trigonometry. Although certain parts of this network may not be immediately available to the student at all times, he always seems able to regenerate them when needed. The identity involving COT and CSC might be one such item.

As you know, it is easy to recognize and distinguish between such students.

The behavior of student A is characterized by inflexibility, even where the student is perfectly able to deal with routine problems on which he has been trained. Student B's behavior, on the other hand, is characterized by relatively greater flexibility, an ability to retrieve or regenerate information as needed in solving a broader variety of tasks.

I should also like to emphasize in passing that all that has been said above applies equally to learning and performance involving concrete materials as well as symbol manipulation. Furthermore, the relationship between syntax and semantics (meaning) can be analyzed in these terms as well.

So far, so good. We have found that what a child learns in a given situation depends on what he already knows. This in itself is not new. The whole concept of prerequisite knowledge is predicated on this idea -- and that goes back almost two decades now. But, we have also learned that the concept of rules which operate on other rules (which are functions defined on functions) is a critical idea that is missing in most existing theories of learning.

We have also learned something about the basic, innate mechanisms by which people seem to learn. While it seems clear that this mechanism has little to do directly with such antiquated ideas as contiguity and reinforcement, the book has not yet been closed on this subject. It is not known, for example, whether the mechanism is characteristic of all people from birth onward, or whether it is only a capability that can be assumed to be available to children, say, of the age of 7 and above. I feel reasonably confident of the latter statement, but hold the former question open.

Now that we have this knowledge, of what value is it to the mathematics teacher? Is it possible to do anything with it? I wish that we could dwell on this topic for I am confident that there is a good deal that can and should be done. Indeed, some of my associates and I have developed and tested, with

prototype materials, systematic techniques whereby given problem domains may be analyzed to determine the underlying rules and higher order rules (e.g., see Scandura, 1972, 1973$_b$). Furthermore, once these rules have been identified, we have shown that they can be taught in a highly efficient manner. In addition, the inclusion of higher order rules has made it possible for our students to solve quite novel problems without explicit training. Whereas transfer has long been a valued goal in educational research, this is, to my knowledge, the that the potential to transfer in predetermined ways to seemingly unrela tasks, has been purposefully, systematically, and in a replicable manner built into instruction.

Figure 4 summarizes the steps used to identify the rules inherent in my text, Mathematics: Concrete Behavioral Foundations (Scandura, 1971).

Insert Figure 4

The result of this analysis was expressed in terms of a large set of tasks, rules, and higher order rules (Ehrenpreis & Scandura, 1974). Some examples are shown in Figure 5.

Insert Figure 5

This method has since been extended with the help of some associates to analyze all of the compass and straight-edge problems described in Polya's Mathematical Discovery (Scandura, Wulfeck, Durnin, & Ehrenpreis, 1974; Scandura, Durnin, Wulfeck, 1974). We have also identified the rules inherent in the old Ball State Algebra Text, which as you know, puts great emphasis on formal proof.

Why would one bother to do this? Is it because I think schooling should

consist of teaching children a set of discrete rules? No. The answer is simply that without such an analysis we only have our intuitions to go on as to what one might reasonably expect a learner to acquire as a result of studying the materials. Once we know what it is we want a child to learn as a result of our analysis, then we will be in a better position, I think, to teach the child that material more efficiently. In addition, for example, suppose we analyze a textbook and we find upon analysis that the underlying rules do not correspond to what our intuition tells us should be taught. The reader might argue that "I can tell that anyway, just by looking at the text," and you probably can. But that misses the point. Having identified exactly what is included in the text, the teacher, or curriculum writer, is in a far better position to change and improve it.

Finally, if the idea is so good, why aren't these approaches being used more broadly? The obvious answer is that these ideas are new. They have not yet had time to filter down to a majority of mathematics educators and teachers. But there is more to it than that. In this process of filtering down, it must be made clear at every turn that the methodology in no way detracts from what mathematics educators and teachers have learned through decades of work with children. Rather, the method is complementary. Once a body of content has been analyzed, it is up to the teacher and to the curriculum writer to determine how that material should best be taught. In effect, science and technology has brought us part of the way, but it is only part way. The teacher and the mathematics educator who work daily on the "firing line" must continue to fill the gap.

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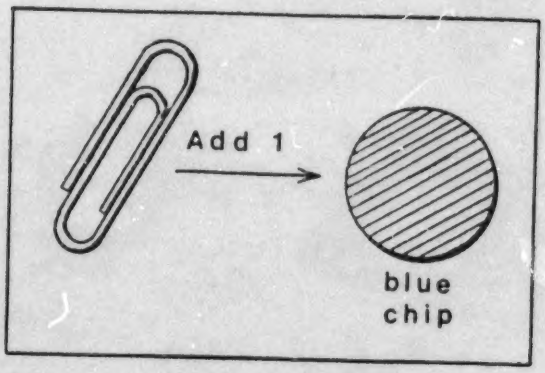
Figure 1



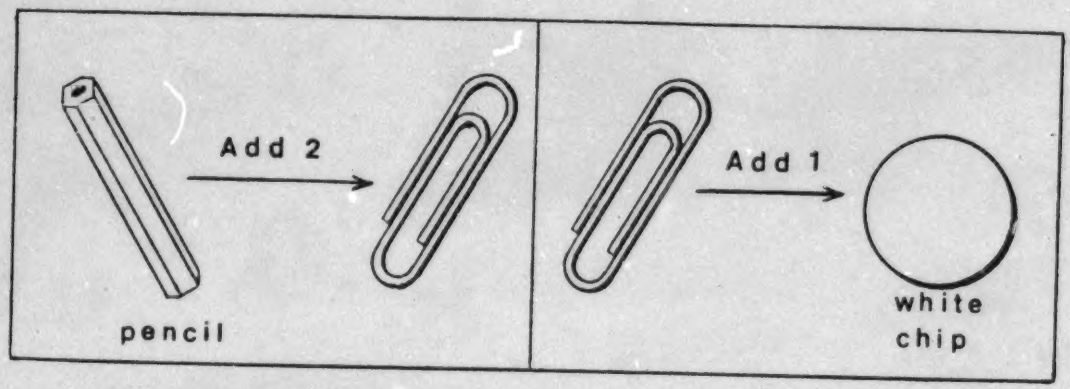


Figure 2

$1 \rightarrow 3$

$2 \rightarrow 6$

$3 \rightarrow 9$

$8 \rightarrow (?)$

Flow diagram for the higher order rule which acts on restricted rules and generates a general rule of the form $n \rightarrow an$.

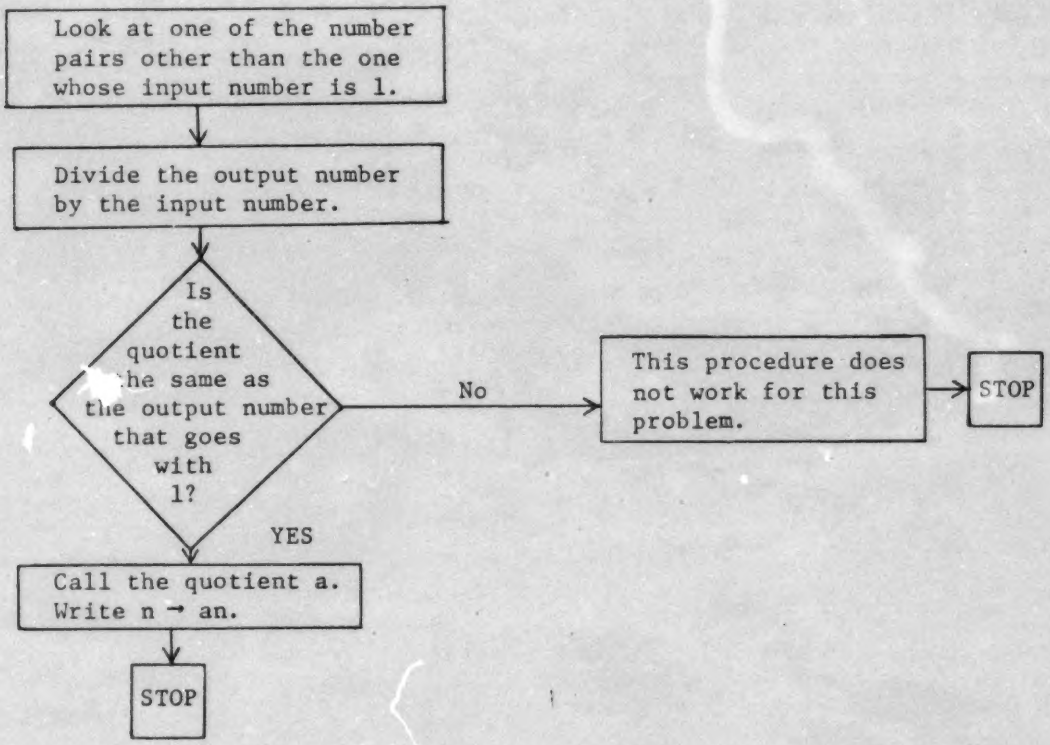

Look at one of the number pairs other than the one whose input number is 1.
Divide the output number by the input number.
Is the quotient the same as the output number that goes with 1?
No
This procedure does not work for this problem.
STOP
YES
Call the quotient a. Write n → an.
STOP

Figure 3

A.

$$\text{SIN}^2\,\theta + \text{COS}^2\,\theta = 1$$

$$\text{TAN}^2\,\theta + 1 = \text{SEC}^2\,\theta$$

$$1 + \text{COT}^2\,\theta = \text{CSC}^2\,\theta$$

B.

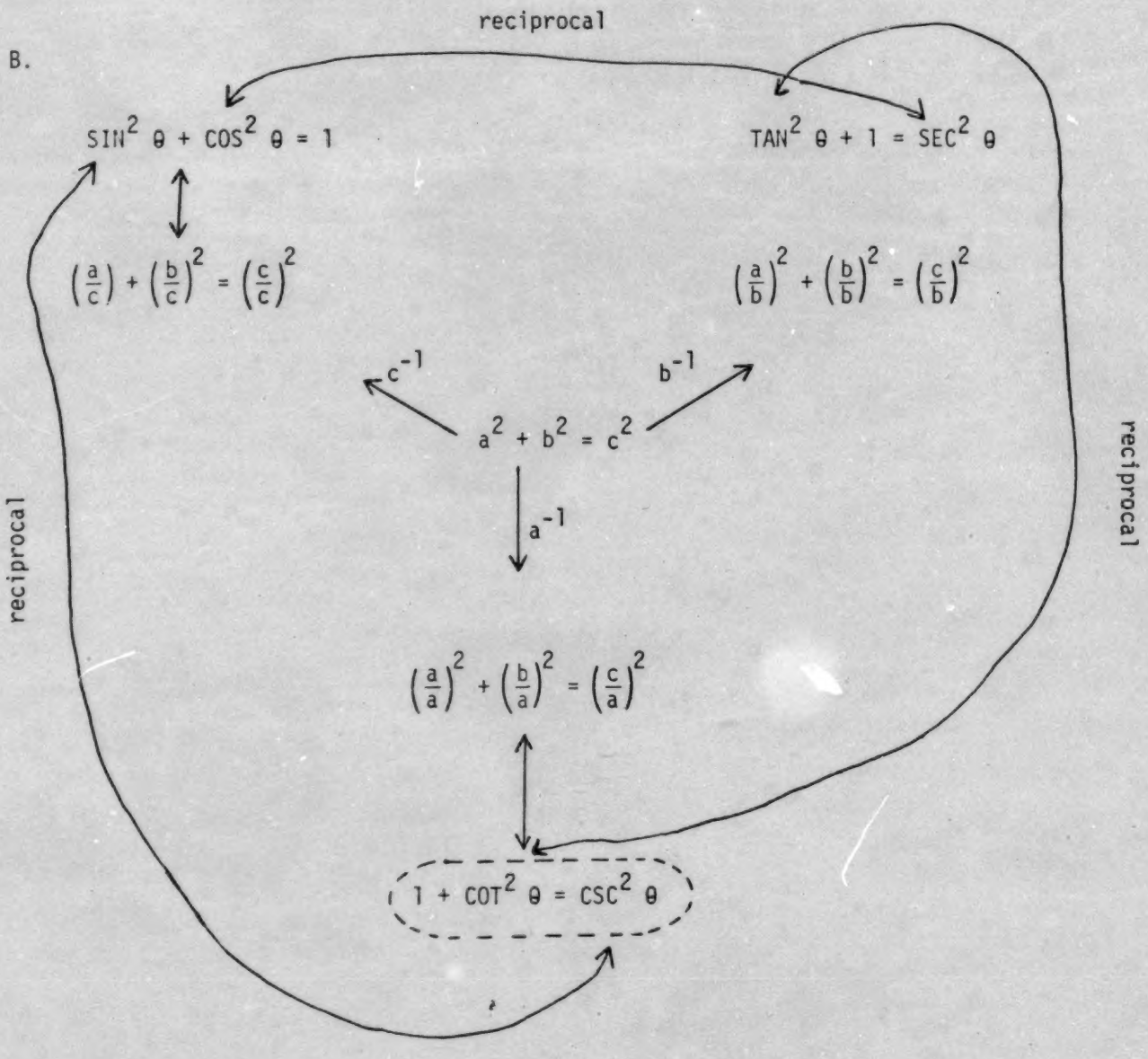


Figure 4

ALGORITHMIC ANALYSIS OF EXISTING TEXTUAL MATERIAL

1. FORMULATE CONTENT PAGE BY PAGE IN TERMS OF BEHAVIORAL OBJECTIVES (i.e., AS TASKS TO BE PERFORMED).

2. DEVISE PROCEDURES FOR SOLVING THOSE TASKS WHICH ARE CONSISTENT WITH THE TEXTUAL CONTENT.

3. LOOK FOR PARALLELS AMONG THE PROCEDURES. IDENTIFY HIGHER ORDER RULES WHICH REFLECT THESE PARALLELS AND ARE AT LEAST NOT INCONSISTENT WITH THE TEXT.

4. ELIMINATE REDUNDANT RULES WHICH CAN BE GENERATED BY APPLICATION OF THE HIGHER ORDER RULES TO OTHER RULES IN THE SET.

5. REPEAT AND REFINE TASKS AND RULES AS NECESSARY.

SAMPLE LOWER AND HIGHER ORDER RULES

TASK A_1: GIVEN A CERTAIN NUMBER OF YARDS, FIND THE EQUIVALENT NUMBER OF FEET.

RULE A_1: MULTIPLY THE NUMBER OF YARDS BY THREE.

TASK A_2: GIVEN A CERTAIN NUMBER OF FEET, FIND THE EQUIVALENT NUMBER OF INCHES.

RULE A_2: MULTIPLY THE NUMBER OF FEET BY TWELVE.

TASK B_1: GIVEN A PIPER CUB STATIONARY ON THE GROUND, GET IT UP TO A CRUISING ALTITUDE.

RULE B_1: ALL THOSE THINGS THAT GO INTO TAKING OFF IN AN AIRPLANE.

TASK B_2: GIVEN A PIPER CUB AT CRUISING ALTITUDE, BRING IT TO A SAFE STOP ON THE GROUND.

RULE B_2: ALL THOSE THINGS THAT GO INTO SAFELY LANDING AN AIRPLANE.

H O TASK: GIVEN TWO RULES SUCH THAT THE OUTPUT OF ONE SERVES AS INPUT FOR THE OTHER, FIND THEIR COMPOSITE (IN LEARNING HIERARCHIES THIS CORRESPONDS TO A "HIGHER ORDER" RULE -- IN ALGORITHMIC ANALYSIS IT IS THE OUTPUT OF THE HIGHER ORDER, H O RULE).

H O RULE: PUT THE TWO RULES TOGETHER SO THAT THE OUTPUT OF ONE ACTS AS INPUT FOR THE OTHER.

TRANSFER TASK A_3: GIVEN A CERTAIN NUMBER OF YARDS, FIND THE EQUIVALENT NUMBER OF INCHES.

TRANSFER TASK B_3: GIVEN A PIPER CUB STATIONARY ON THE GROUND, "SOLO."